

A NEW PROOF OF SEMICIRCLE LAW OF FIXED TRACE SQUARE ENSEMBLE

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ABSTRACT. In the present paper, we give a simple proof of the level density of fixed trace square ensemble. We derive the integral equation of the level density of fixed trace square ensemble. Then we analyze the asymptotic behavior of the level density.

1. INTRODUCTION

E.P.Wigner in [11] and [12] showed that the level density (one point correlation) of "border matrix" ensemble which is a kind of real symmetric matrix ensemble, asymptotically approaches to the semicircle, i.e.,

$$(1.1) \quad \sigma_{SY.S,N}(x) \rightarrow \sqrt{2N - x^2}/\pi.$$

Porter and Rosenzweig also found that the level density of some special random matrices satisfy semi-circle law [7][8].

Wigner's motivation arose from the consideration of properties of wave functions of quantum mechanical system that are extremely complicated that statistical considerations can be applied to them. From a computational point of view, random matrix theory, as a statistical theory, is very important to describe extremely complicated quantum mechanical systems such as large atoms, QCD physics (see [10] for review). It is instructive to continue to work on the properties of RMT in the problems of complicated quantum systems.

Recently there has been work on the analysis of top eigenvalues of random matrices ([13]). When we evaluate some quantities, for example the ratio of the expectation of the largest square of eigenvalues and the expectation of the square of an arbitrary eigenvalues

$$(1.2) \quad \frac{\langle \max_{1 \leq i \leq N} x_i^2 \rangle_{GUE}}{\langle x_1^2 \rangle_{GUE}} = \frac{\int \cdots \int \max_{1 \leq i \leq N} x_i^2 e^{-\sum_1^N x_i^2} \prod_{i < j} (x_i - x_j)^2 dx_1 dx_2 \cdots dx_N}{\int \cdots \int \max_{1 \leq i \leq N} x_i^2 e^{-\sum_1^N x_i^2} \prod_{i < j} (x_i - x_j)^2 dx_1 dx_2 \cdots dx_N}$$

, we can change the integrals into polar coordinates. We can see the integration on vandermonde space is the difficulty. Therefore it is important to consider the "fixed trace square ensemble" on \mathbb{R}^N

$$(1.3) \quad P_{v,N}(x_1, x_2, \cdots, x_N) = C_v \prod_{i < j} |x_i - x_j|^2,$$

on the $N-1$ dimensional unit sphere $\{(x_1, \cdots, x_N); \sum_1^N x_i^2 = 1\}$ and where C_v is a normalization constant. Stieltjes in 1914 proved a theorem that states the maximum value of vandermonde polynomial $\prod_{i,j} |x_i - x_j|^2$ in the ball $\{x_1^2 + x_2^2 + \cdots + x_N^2 \leq$

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$N(N-1)\}$ achieves its maximal values at zeros of Hermite polynomials. In [13], the author considered the bounded trace square ensemble and obtained semicircle law of bounded trace square ensemble. In [?], the authors reobtained the semicircle law of fixed trace square ensemble by comparing it with GUE and saddle point method. In the present paper, we would like to give a simple proof of the semicircle law of fixed square ensemble.

2. INTEGRALS OF FIXED TRACE SQUARE ENSEMBLE

Let's give a general definition of the ensemble on a homogeneous space.

Definition 1. $P(X) = P(x_1, x_2, \dots, x_N)$ is a probability distribution satisfying on a homogenous space M with respect to the permutation group action on x_1, x_2, \dots, x_N , which are the local coordinates. The level density is defined to be

$$(2.1) \quad \sigma_P(x) = N \int P(x_1, x_2, \dots, x_N) dx_2 \cdots dx_N.$$

In this paper, we are concerned with the case that M is the $N-1$ dimensional sphere.

We would like to compute the level density of the ensemble $P(x_1, x_2, \dots, x_N) = C_N \prod_{i < j} |x_i - x_j|^2$ on the N dimensional ball $B_1 = \{(x_1, x_2, \dots, x_N); \sum_{i=1}^N x_i^2 < 1\}$. The level density on sphere will easily follow. We will use Selberg's integral [9].

Theorem 2.1. (Selberg) For any positive integer n , let $dx \equiv dx_1 \cdots dx_N$,

$$(2.2) \quad \Delta(x) \equiv \Delta(x_1, \dots, x_n) = \prod_{1 \leq j < l \leq n} (x_j - x_l), \text{ if } n > 1, \\ \Delta(x) = 1, \text{ if } n = 1,$$

and

$$(2.3) \quad \Phi(x) \equiv \Phi(x_1, \dots, x_N) = |\Delta(x)|^{2\gamma} \prod_{j=1}^N x_j^{\alpha-1} (1-x_j)^{\beta-1}.$$

Then

$$(2.4) \quad I(\alpha, \beta, \gamma, n) \equiv \int_0^1 \cdots \int_0^1 \Phi(x) dx = \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha_j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(n+j-1)\gamma)},$$

and for $1 \leq m \leq n$,

$$(2.5) \quad \int_0^1 \cdots \int_0^1 x_1 x_2 \cdots x_m \Phi(x) dx = \prod_{j=1}^m \frac{\alpha + (n-j)\gamma}{\alpha + \beta + (2n-j-1)\gamma} \int_0^1 \cdots \int_0^1 \Phi(x) dx,$$

valid for integer n and complex α, β, γ with

$$(2.6) \quad \operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \gamma > -\min\left(\frac{1}{n}, \frac{\operatorname{Re} \alpha}{n-1}, \frac{\operatorname{Re} \beta}{n-1}\right).$$

We need to compute the constant in the expression of the ensemble. We cannot directly apply Selberg's theorem, because the domain we integrate on is the unit ball instead of \mathbb{R}^N . By the virtue of gamma function, we can change the integral on the unit ball to the integral on \mathbb{R}^N of Selberg's type. Let us prove the following identity first.

Proposition 1. *For any $R > 0$, we have*

$$(2.7) \quad \int_{\sum_{i=1}^N x_i^2 \leq R^2} \Delta^2 dx = R^{n^2} \frac{1}{\Gamma(n^2/2 + 1)} (2\pi)^{n/2} 2^{-N^2/2} \prod_{j=1}^N \Gamma(1 + j)$$

Proof. In Selberg's integral, letting $x_i = \frac{y_i}{2L} + \frac{1}{2}[6]$ and $L \rightarrow \infty$, then

$$(2.8) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta|^{2\gamma} \prod_{j=1}^N e^{-ax_j^2} dx_j = (2\pi)^{n/2} (2a)^{-N(\gamma(n-1)+1)/2} \prod_{j=1}^N \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}$$

Multiplying $a^{\beta-1}e^{-a}$ to (2.8) and integrating with respect to a , we get

$$(2.9) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta|^{2\gamma} \prod_{j=1}^N (1 + \sum_{i=1}^N x_i^2)^{-\beta} dx_j$$

$$(2.10) \quad = \frac{\Gamma(\beta - N(\gamma(n-1) + 1)/2)}{\Gamma(\beta)} (2\pi)^{n/2} 2^{-N(\gamma(n-1)+1)/2} \prod_{j=1}^N \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}.$$

We change variables by $x_i = \frac{y_i}{\sqrt{1 - \sum_{j=1}^N y_j^2}}$. The determinant is

$$(2.11) \quad \frac{r_x^{n-1} dr_x}{r_y^{n-1} dr_y} = \frac{1}{(1 - \sum_{j=1}^N y_j^2)^{\frac{n-1}{2}}} \frac{1}{(1 - \sum_{j=1}^N y_j^2)^{\frac{3}{2}}} = \frac{1}{(1 - \sum_{j=1}^N y_j^2)^{\frac{n}{2}+1}},$$

where r_x, r_y are the radius coordinates in the polar coordinate system. Therefore we have

$$(2.12) \quad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\Delta|^{2\gamma} (1 + \sum_{i=1}^N x_i^2)^{-\beta} \prod_{j=1}^N dx_j$$

$$(2.13) \quad = \int_{\sum_{j=1}^N y_j^2 \leq 1} |\Delta(y)|^{2\gamma} (1 - \sum_{i=1}^N y_i^2)^{\beta - N(\gamma(n-1)+1)/2 - 1} \prod_{j=1}^N dy_j.$$

Then we end up with

$$(2.14) \quad \int_{\sum_{j=1}^N y_j^2 \leq 1} |\Delta(y)|^{2\gamma} (1 - \sum_{i=1}^N y_i^2)^{\beta - N(\gamma(n-1)+1)/2 - 1} \prod_{j=1}^N dy_j$$

$$= \frac{\Gamma(\beta - N(\gamma(n-1) + 1)/2)}{\Gamma(\beta)} (2\pi)^{n/2} 2^{-N(\gamma(n-1)+1)/2} \prod_{j=1}^N \frac{\Gamma(1+j\gamma)}{\Gamma(1+\gamma)}.$$

Now set $\gamma = 1$ and $\beta - N(\gamma(n-1) + 1)/2 - 1 = 0$. We get

$$(2.15) \quad \int_{\sum_{j=1}^N y_j^2 \leq 1} |\Delta(y)|^2 \prod_{j=1}^N dy_j = \frac{1}{\Gamma(n^2/2 + 1)} (2\pi)^{n/2} 2^{-N^2/2} \prod_{j=1}^N \Gamma(1 + j).$$

By rescaling, we conclude the result of this proposition. \square

we are now ready to prove the following identity.

Theorem 2.2. *The level density of $P(x) = \text{const.} \Delta^2(x)$ in the unit ball is*

Proof. The level density of $P(x)$ is

$$\begin{aligned}
\sigma_P(y_1) &= N \int \cdots \int_{y_2^2 + y_3^2 + \cdots + y_N^2 \leq 1 - y^2} dy_2 \cdots dy_N \Delta^2(y) \\
&= N \int \cdots \int_{y_2^2 + y_3^2 + \cdots + y_N^2 \leq 1 - y^2} dy_2 \cdots dy_N \prod_{j=2}^N (y_1 - y_j)^2 \prod_{2 \leq i \leq j \leq N} (y_i - y_j)^2 \\
&= N \int \cdots \int_{y_2^2 + y_3^2 + \cdots + y_N^2 \leq 1 - y^2} dy_2 \cdots dy_N \prod_{j=2}^N (y_1^2 - 2y_1 y_j + y_j^2) \prod_{2 \leq i \leq j \leq N} (y_i - y_j)^2 \\
&= N \int \cdots \int_{y_2^2 + y_3^2 + \cdots + y_N^2 \leq 1 - y^2} dy_2 \cdots dy_N \left(\sum_{p=0}^{N-1} \sum_{q=0}^{N-1-p} y_1^p ((-2)^q y_1^q) \right) \prod_{2 \leq i \leq j \leq N} (y_i - y_j)^2 \\
(2.16) \quad &= N \left(\sum_{p=0}^{N-1} \sum_{q=0}^{N-1-p} (-2)^q y_1^p + q(1 - y_1^2)^{1/2(-1-2p-q+N^2)} \langle y_2 y_3 \cdots y_{q+1} y_{q+2}^2 y_{q+3}^2 \cdots y_{N-p}^2 \rangle_{P, B_1} \right)
\end{aligned}$$

□

One can easily apply Aomoto's technique ([1]) to compute $\langle x_1 x_2 \cdots x_m \rangle$. However, we wish that we could get an analytic expression of the level density. Unfortunately, as we have seen the above theorem, we have to explicitly write expectations such as $\langle x_1^2 x_2^2 \cdots x_m^2 x_{m+1} x_{m+2} \cdots x_n \rangle$ which turns out to be very hard and remains an open problem. This fact makes us have to think about the numerical solution of the level density.

3. THE INTEGRAL EQUATION OF LEVEL DENSITY

We would like to derive the integral equation of the level density of fixed trace square ensemble in another way. The idea is this. Consider the level density of GUE. On the sphere with radius r , the level density is $\frac{1}{r} \Phi(\frac{x}{r})$ with weight $C_N^{-1} e^{-r^2} r^{n-1} r^{n(n-1)}$ where

$$(3.1) \quad C_N = \int_0^\infty e^{-r^2} r^{N-1} r^{N(N-1)} dr = \frac{1}{2} \Gamma\left(\frac{N^2}{2}\right).$$

So we have

Theorem 3.1. *The level density of fixed trace square ensemble satisfies the integral equation*

$$(3.2) \quad 2\Gamma^{-1}\left(\frac{N^2}{2}\right) \int_x^\infty e^{-r^2} r^{N^2-2} \sigma_{\Phi, n}(x/r) dr = \sigma_{GUE, n}(x).$$

Proof. For any $f(x_1) \in C_0^\infty(\mathbb{R})$, we change the following integral into polar coordinate sytem.

$$\begin{aligned}
& \int \sigma_{GUE}(x_1) f(x_1) dx_1 \\
&= N \int C_{GUE,N} e^{-\sum_1^N x_i^2} \prod_{i < j} (x_i - x_j)^2 f(x_1) dx_1 dx_2 \cdots dx_N \\
&= N \int_0^\infty e^{-r^2} r^{N-1+N(N-1)} dr \int_\Omega \Delta^2(\Omega) f(r\Omega_1) d\Omega \\
&= \int_0^\infty e^{-r^2} r^{N-1+N(N-1)} dr \int_{-1}^1 d\Omega_1 \sigma_v(\Omega_1) f(r\Omega_1) \\
&= \text{const.} \int_0^\infty e^{-r^2} r^{N-1+N(N-1)-1} dr \int_{-r}^r dx_1 \sigma_v(x_1/r) f(x_1) \\
(3.3) \quad &= 2\Gamma^{-1} \int_{|x_1|}^\infty e^{-r^2} r^{N-1+N(N-1)-1} dr \int_{-\infty}^\infty dx_1 \sigma_v(x_1/r) f(x_1).
\end{aligned}$$

The constant in (3.3) is computed by Proposition 1. \square

Similarly, we can derive the integral equation of two point cluster function of fixed trace square ensemble $Y_{v,2}(x_1, x_2)$ in terms of the cluster function of GUE $Y_{GUE,2}(x_1, x_2) = (\frac{\sin \pi(x_1-x_2)}{\pi(x_1-x_2)})^2$.

It immediately follows that the asymptotic behavior of $\sigma_v(0)$.

Corollary 1.

$$(3.4) \quad \sigma_{v,N}(0) = \frac{\sqrt{2N}\Gamma(N^2/2)}{\pi\Gamma(N^2/2 - 1/2)}(1 + o(1)).$$

4. MAIN THEOREM

It is well known that the eigenvalue distribution of GUE, $\rho(x) = \sqrt{\frac{2}{N}}\sigma_{GUE,N}(\sqrt{2N}x)$ goes to semicircle:

$$(4.1) \quad \rho(x) \rightarrow S(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } |x| \leq 1 \\ 0, & \text{else.} \end{cases}$$

Theorem 4.1. *It is possible to prove the asymptotic behavior of the integral equation (3.2). In fact, The normalized eigenvalue distribution of fixed trace square ensemble $\rho_{v,N}(x) = \frac{2}{N^{\frac{3}{2}}}\sigma_{v,N}(\frac{2x}{\sqrt{N}})$ goes to the semicircle (4.1).*

Proof. First we have to estimate the L^∞ norm of σ_N .

Define a function

$$(4.2) \quad \xi(x) = \inf\{\xi; \sigma_{v,N}(\frac{x}{\xi}) 2\Gamma^{-1}(N^2/2) \int_{|x|}^\infty e^{-r^2} r^{N^2-2} dr\}$$

Note that $\xi(x)$ is a continuous function on $[0, \infty)$. Then $\frac{x}{\xi(x)}$ (at $x = 0$, this function is defined to be zero) is also a continuous function. Since $\sigma_{GUE,N} \rightarrow 0$ ([4]) as $x \rightarrow \infty$, then

$$(4.3) \quad \sigma_{v,N}(x/\xi(x)) = \frac{\sigma_{GUE,N}(x)}{2\Gamma^{-1}(N^2/2) \int_{|x|}^\infty e^{-r^2} r^{N^2-2} dr} \rightarrow 0,$$

as $x \rightarrow \infty$ Therefore $\sigma_{v,N}(\frac{x}{\xi(x)}) \rightarrow 0$ and $\frac{x}{\xi(x)} \rightarrow 1$. By the expansion of hermite functions in [4], we know $\forall x \in [0, 1]$,

$$(4.4) \quad \frac{\sigma_{v,N}(x)}{2\Gamma^{-1}(N^2/2) \int_0^\infty e^{-r^2} r^{N^2-2} dr} \leq CN^{\frac{1}{2}}.$$

Therefore

$$(4.5) \quad \sigma_{v,N} \leq CN^{\frac{3}{2}}.$$

We shall use the following lemma:

Lemma 4.2. $\forall 0 < \alpha < 1, \forall m > 0$, if N is sufficiently large,

$$(4.6) \quad 1 - N^{-m} \leq \frac{\int_{N/\sqrt{2}-N\alpha}^{N/\sqrt{2}+N\alpha} e^{-r^2} r^{N^2-2} dr}{\int_0^\infty e^{-r^2} r^{N^2-2} dr} \leq 1.$$

The proof is straightforward.

For $\forall \alpha > 0$, if N is sufficiently large,

$$\begin{aligned} & 2\Gamma^{-1}(N^2/2) \int_{|x|} e^{-r^2} r^{N^2-2} \sigma_{v,N}(x/r) dr \\ &= 2\Gamma^{-1}(N^2/2) \left(\int_{|x|}^{N/\sqrt{2}-N\alpha} + \int_{N/\sqrt{2}-N\alpha}^{N/\sqrt{2}+N\alpha} + \int_{N/\sqrt{2}+N\alpha}^\infty \right) e^{-r^2} r^{N^2-2} \sigma_{v,N}(x/r) dr \\ &= 2\Gamma^{-1}(N^2/2) \left(\int_{|x|}^{N/\sqrt{2}-N\alpha} + \int_{N/\sqrt{2}+N\alpha}^\infty \right) e^{-r^2} r^{N^2-2} N^{1.5} dr + \int_{N/\sqrt{2}-N\alpha}^{N/\sqrt{2}+N\alpha} e^{-r^2} r^{N^2-2} \sigma_{v,N}(x/r) dr \\ &= 2\Gamma^{-1}(N^2/2) \int_{N/\sqrt{2}-N\alpha}^{N/\sqrt{2}+N\alpha} e^{-r^2} r^{N^2-2} \sigma_{v,N}(x/r) dr + O(1/N^5) \\ (4.7) \quad &= 2\Gamma^{-1}(N^2/2) \sigma_{v,N}(x/r(x)) \int_{N/\sqrt{2}-N\alpha}^{N/\sqrt{2}+N\alpha} e^{-r^2} r^{N^2-2} dr + O(1/N^5), \end{aligned}$$

where $N/\sqrt{2} - N\alpha \leq r(x) \leq N/\sqrt{2} + N\alpha$. Therefore $\forall 0 \leq y$,

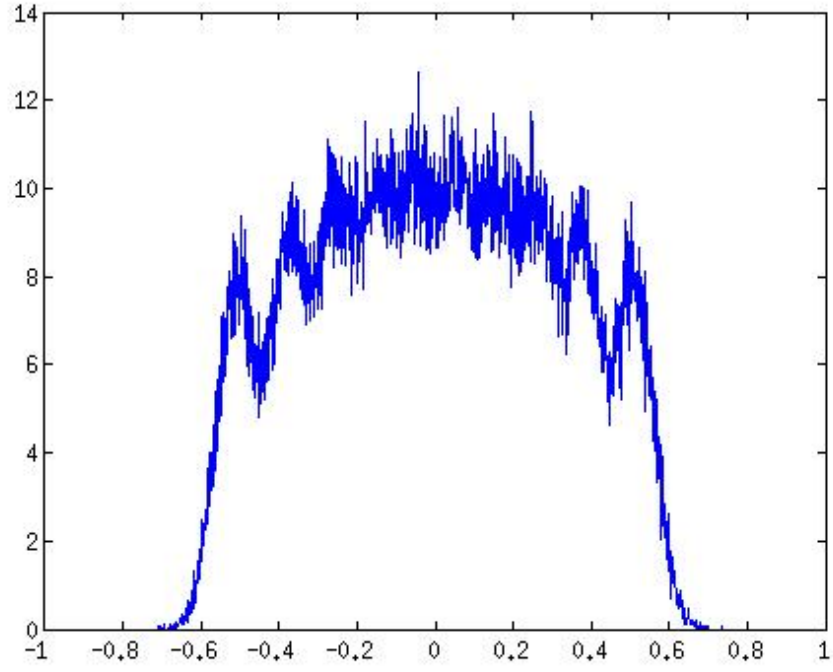
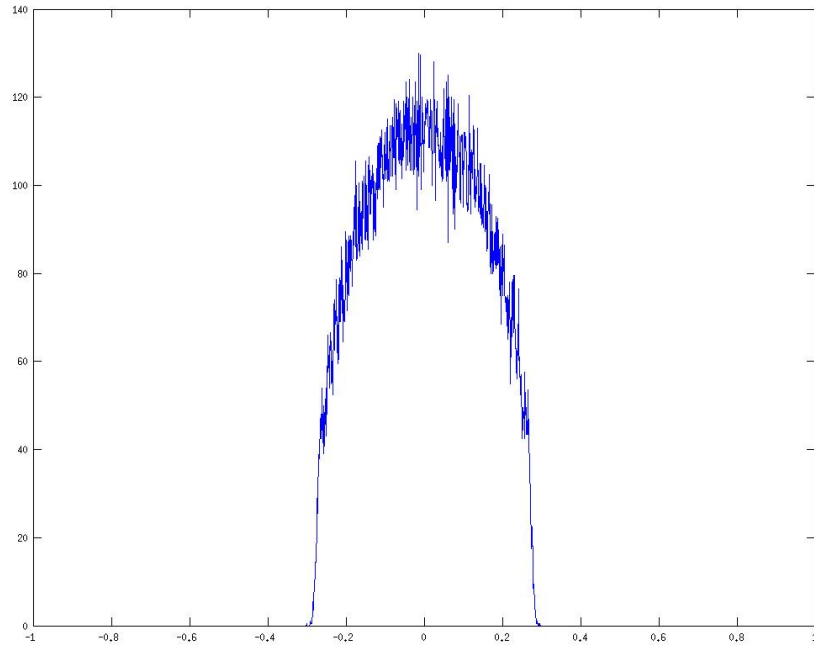
$$(4.8) \quad \sigma_{v,N}(y) = \frac{\Gamma(N^2/2)}{2 \int_{N/\sqrt{2}-N\alpha}^{N/\sqrt{2}+N\alpha} e^{-r^2} r^{N^2-2} dr} \sigma_{GUE,N}(r(\tilde{y})y) + O(1/N^4).$$

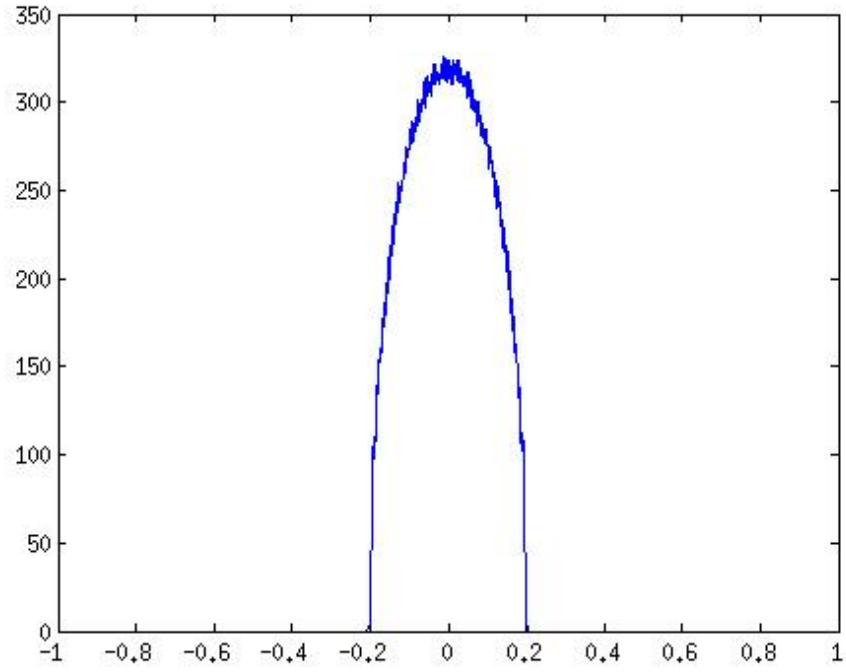
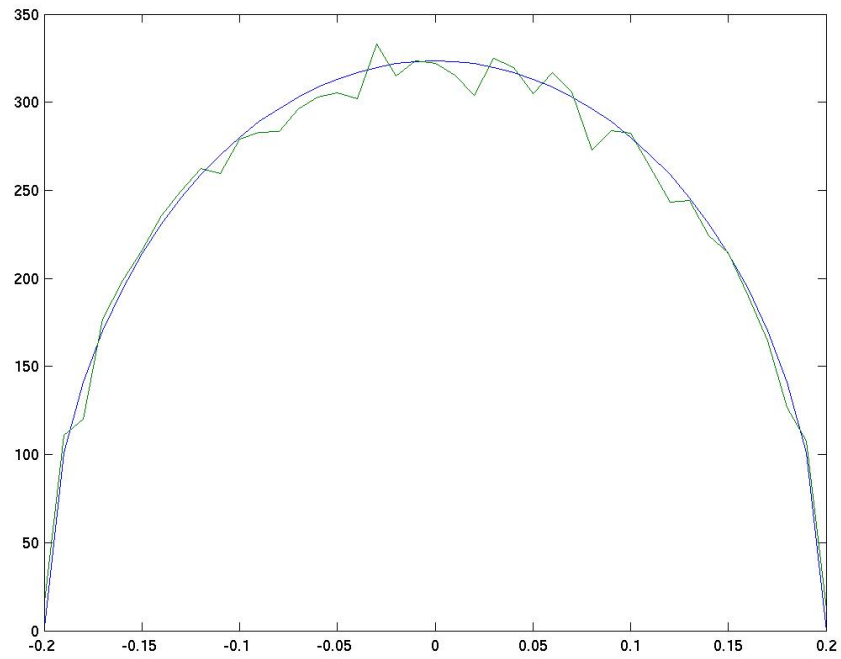
We notice that $\frac{\Gamma(N^2/2)}{2 \int_0^\infty e^{-r^2} r^{N^2-2} dr} \rightarrow N/\sqrt{2} + O(1/\sqrt{N})$ (see [5]). Therefore,

$$\begin{aligned}
& |\rho_{v,N}(y) - S(y)| \\
& \leq \left| \frac{2}{N^{\frac{3}{2}}} \sigma_{v,N}\left(\frac{2y}{\sqrt{N}}\right) - S(y) \right| \\
& \leq \left| \sqrt{\frac{2}{N}} \sigma_{GUE,N}(\tilde{r}2y/\sqrt{N}) - S(y) \right| + o(1) \\
& \leq \left| \sqrt{\frac{2}{N}} \sigma_{GUE,N}(\sqrt{2N}(\tilde{r}2y/(\sqrt{N}\sqrt{2N}))) - S(\tilde{r}2y/(\sqrt{N}\sqrt{2N})) \right| + |S(\tilde{r}2y/(\sqrt{N}\sqrt{2N})) - S(y)| + o(1) \\
& \stackrel{(4.9)}{\leq} \sup_{|x-y| \leq \alpha} |S(x) - S(y)| + o(1)
\end{aligned}$$

□

We use the software Matlab to estimate the level density $\sigma_{\Phi,N}$ of fixed trace square ensemble in the case $N = 10$, $N = 50$, and $N = 100$. Surprisingly, these graphs look really like semicircles.

FIGURE 1. Level density of fixed trace square ensemble $N=10$ FIGURE 2. Level density of fixed trace square ensemble $N=50$

FIGURE 3. Level density of fixed trace square ensemble $N=100$ FIGURE 4. the coincidence of the semicircle and the level density when $N = 100$

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